8.2 Exercises

VOCABULARY CHECK:

In Exercises 1–4, fill in the blanks.

1. Two matrices are ________ if all of their corresponding entries are equal.
2. When performing matrix operations, real numbers are often referred to as ________.
3. A matrix consisting entirely of zeros is called a ________ matrix and is denoted by ________.
4. The $n \times n$ matrix consisting of 1's on its main diagonal and 0's elsewhere is called the ________ matrix of order n.

In Exercises 5 and 6, match the matrix property with the correct form. *A*, *B*, and *C* are matrices of order $m \times n$, and *c* and *d* are scalars.

5. (a) $1A = A$ — (i) Distributive Property
   (b) $A + (B + C) = (A + B) + C$ — (ii) Commutative Property of Matrix Addition
   (c) $(c + d)A = cA + dA$ — (iii) Scalar Identity Property
   (d) $(cd)A = c(dA)$ — (iv) Associative Property of Matrix Addition
   (e) $A + B = B + A$ — (v) Associative Property of Scalar Multiplication
6. (a) $A + O = A$ — (i) Distributive Property
   (b) $c(AB) = A(cB)$ — (ii) Additive Identity of Matrix Addition
   (c) $A(B + C) = AB + AC$ — (iii) Associative Property of Multiplication
   (d) $A(BC) = (AB)C$ — (iv) Associative Property of Scalar Multiplication

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, find *x* and *y*.

1. $\begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$

2. $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

4. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

In Exercises 5–12, if possible, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.

5. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix}$,
   $B = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ -3 & 4 & 9 & -6 & -7 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 2 \\ 5 & 4 & -1 \\ 0 & 8 & -6 \\ -4 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -4 & -7 \\ 10 & -9 & -1 \\ 3 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$

11. $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$

In Exercises 13–18, evaluate the expression.

13. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

Example 12 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors.

	Women's Team	Men's Team
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$80, each ball costs \$6, and each glove costs \$60. Use matrices to find the total cost of equipment for each team.

Solution

The equipment lists E and the costs per item C can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

and

$$C = [80 \quad 6 \quad 60].$$

The total cost of equipment for each team is given by the product

$$\begin{aligned} CE &= [80 \quad 6 \quad 60]\begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= [80(12) + 6(45) + 60(15) \quad 80(15) + 6(38) + 60(17)] \\ &= [2130 \quad 2448]. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2130 and the total cost of equipment for the men's team is \$2448. Notice that you cannot find the total cost using the product EC because EC is not defined. That is, the number of columns of E (2 columns) does not equal the number of rows of C (1 row).

✓CHECKPOINT Now try Exercise 63.

Writing About Mathematics

Problem Posing Write a matrix multiplication application problem that uses the matrix

$$A = \begin{bmatrix} 20 & 42 & 33 \\ 17 & 30 & 50 \end{bmatrix}.$$

Exchange problems with another student in your class. Form the matrices that represent the problem, and solve the problem. Interpret your solution in the context of the problem. Check with the creator of the problem to see if you are correct. Discuss other ways to represent and/or approach the problem.

15. $4\left(\begin{bmatrix}-4&0&1\\0&2&3\end{bmatrix}-\begin{bmatrix}2&1&-2\\3&-6&0\end{bmatrix}\right)$

16. $\frac{1}{2}([5\quad -2\quad 4\quad 0]+[14\quad 6\quad -18\quad 9])$

17. $-3\left(\begin{bmatrix}0&-3\\7&2\end{bmatrix}+\begin{bmatrix}-6&3\\8&1\end{bmatrix}\right)-2\begin{bmatrix}4&-4\\7&-9\end{bmatrix}$

18. $-1\begin{bmatrix}4&11\\-2&-1\\9&3\end{bmatrix}+\frac{1}{6}\left(\begin{bmatrix}-5&-1\\3&4\\0&13\end{bmatrix}+\begin{bmatrix}7&5\\-9&-1\\6&-1\end{bmatrix}\right)$

In Exercises 19–22, use the matrix capabilities of a graphing utility to evaluate the expression. Round your results to three decimal places, if necessary.

19. $\frac{3}{7}\begin{bmatrix}2&5\\-1&-4\end{bmatrix}+6\begin{bmatrix}-3&0\\2&2\end{bmatrix}$

20. $55\left(\begin{bmatrix}14&-11\\-22&19\end{bmatrix}+\begin{bmatrix}-22&20\\13&6\end{bmatrix}\right)$

21. $-\begin{bmatrix}3.211&6.829\\-1.004&4.914\\0.055&-3.889\end{bmatrix}-\begin{bmatrix}-1.630&-3.090\\5.256&8.335\\-9.768&4.251\end{bmatrix}$

22. $-12\left(\begin{bmatrix}6&20\\1&-9\\-2&5\end{bmatrix}+\begin{bmatrix}14&-15\\-8&-6\\7&0\end{bmatrix}+\begin{bmatrix}-31&-19\\16&10\\24&-10\end{bmatrix}\right)$

In Exercises 23–26, solve for *X* in the equation, given

$$A=\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}0&3\\2&0\\-4&-1\end{bmatrix}.$$

23. $X=3A-2B$

24. $2X=2A-B$

25. $2X+3A=B$

26. $2A+4B=-2X$

In Exercises 27–34, if possible, find *AB* and state the order of the result.

27. $A=\begin{bmatrix}2&1\\-3&4\\1&6\end{bmatrix},\quad B=\begin{bmatrix}0&-1&0\\4&0&2\\8&-1&7\end{bmatrix}$

28. $A=\begin{bmatrix}1&0&3&-2\\6&13&8&-17\end{bmatrix},\quad B=\begin{bmatrix}1&6\\4&2\end{bmatrix}$

29. $A=\begin{bmatrix}0&-1&0\\4&0&2\\8&-1&7\end{bmatrix},\quad B=\begin{bmatrix}2&1\\-3&4\\1&6\end{bmatrix}$

30. $A=\begin{bmatrix}-1&3\\4&-5\\0&2\end{bmatrix},\quad B=\begin{bmatrix}1&2\\0&7\end{bmatrix}$

31. $A=\begin{bmatrix}1&0&0\\0&4&0\\0&0&-2\end{bmatrix},\quad B=\begin{bmatrix}3&0&0\\0&-1&0\\0&0&5\end{bmatrix}$

32. $A=\begin{bmatrix}5&0&0\\0&-8&0\\0&0&7\end{bmatrix},\quad B=\begin{bmatrix}\frac{1}{5}&0&0\\0&-\frac{1}{8}&0\\0&0&\frac{1}{2}\end{bmatrix}$

33. $A=\begin{bmatrix}0&0&5\\0&0&-3\\0&0&4\end{bmatrix},\quad B=\begin{bmatrix}6&-11&4\\8&16&4\\0&0&0\end{bmatrix}$

34. $A=\begin{bmatrix}10\\12\end{bmatrix},\quad B=\begin{bmatrix}6&-2&1&6\end{bmatrix}$

In Exercises 35–40, use the matrix capabilities of a graphing utility to find *AB*, if possible.

35. $A=\begin{bmatrix}5&6&-3\\-2&5&1\\10&-5&5\end{bmatrix},\quad B=\begin{bmatrix}1&-1&2\\8&1&4\\4&-2&9\end{bmatrix}$

36. $A=\begin{bmatrix}11&-12&4\\14&10&12\\6&-2&9\end{bmatrix},\quad B=\begin{bmatrix}12&10\\-5&12\\15&16\end{bmatrix}$

37. $A=\begin{bmatrix}-3&8&-6&8\\-12&15&9&6\\5&-1&1&5\end{bmatrix},\quad B=\begin{bmatrix}3&1&6\\24&15&14\\16&10&21\\8&-4&10\end{bmatrix}$

38. $A=\begin{bmatrix}-2&4&8\\21&5&6\\13&2&6\end{bmatrix},\quad B=\begin{bmatrix}2&0\\-7&15\\32&14\\0.5&1.6\end{bmatrix}$

39. $A=\begin{bmatrix}9&10&-38&18\\100&-50&250&75\end{bmatrix},$

$B=\begin{bmatrix}52&-85&27&45\\40&-35&60&82\end{bmatrix}$

40. $A=\begin{bmatrix}15&-18\\-4&12\\-8&22\end{bmatrix},\quad B=\begin{bmatrix}-7&22&1\\8&16&24\end{bmatrix}$

In Exercises 41–46, if possible, find (a) *AB*, (b) *BA*, and (c) A^2. (*Note:* $A^2=AA$.)

41. $A=\begin{bmatrix}1&2\\4&2\end{bmatrix},\quad B=\begin{bmatrix}2&-1\\-1&8\end{bmatrix}$

42. $A=\begin{bmatrix}2&-1\\1&4\end{bmatrix},\quad B=\begin{bmatrix}0&0\\3&-3\end{bmatrix}$

43. $A=\begin{bmatrix}3&-1\\1&3\end{bmatrix},\quad B=\begin{bmatrix}1&-3\\3&1\end{bmatrix}$

44. $A=\begin{bmatrix}1&-1\\1&1\end{bmatrix},\quad B=\begin{bmatrix}1&3\\-3&1\end{bmatrix}$

45. $A=\begin{bmatrix}7\\8\\-1\end{bmatrix},\quad B=\begin{bmatrix}1&1&2\end{bmatrix}$

46. $A=\begin{bmatrix}3&2&1\end{bmatrix},\quad B=\begin{bmatrix}2\\3\\0\end{bmatrix}$

In Exercises 47–50, evaluate the expression. Use the matrix capabilities of a graphing utility to verify your answer.

47. $\begin{bmatrix}3&1\\0&-2\end{bmatrix}\begin{bmatrix}1&0\\-2&2\end{bmatrix}\begin{bmatrix}1&0\\2&4\end{bmatrix}$

48. $-3\left(\begin{bmatrix} 6 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & -3 \\ 4 & 1 \end{bmatrix}\right)$

49. $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 4 & 0 \\ 0 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -3 & 5 \\ 0 & -3 \end{bmatrix}\right)$

50. $\begin{bmatrix} 3 \\ -1 \\ 5 \\ 7 \end{bmatrix}([5 \quad -6] + [7 \quad -1] + [-8 \quad 9])$

In Exercises 51–58, (a) write the system of linear equations as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix X.

51. $\begin{cases} -x_1 + x_2 = 4 \\ -2x_1 + x_2 = 0 \end{cases}$

52. $\begin{cases} 2x_1 + 3x_2 = 5 \\ x_1 + 4x_2 = 10 \end{cases}$

53. $\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$

54. $\begin{cases} -4x_1 + 9x_2 = -13 \\ x_1 - 3x_2 = 12 \end{cases}$

55. $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 - x_3 = -6 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

56. $\begin{cases} x_1 + x_2 - 3x_3 = 9 \\ -x_1 + 2x_2 = 6 \\ x_1 - x_2 + x_3 = -5 \end{cases}$

57. $\begin{cases} x_1 - 5x_2 + 2x_3 = -20 \\ -3x_1 + x_2 - x_3 = 8 \\ -2x_2 + 5x_3 = -16 \end{cases}$

58. $\begin{cases} x_1 - x_2 + 4x_3 = 17 \\ x_1 + 3x_2 = -11 \\ -6x_2 + 5x_3 = 40 \end{cases}$

59. ***Manufacturing*** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. The number of units of guitars produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}.$$

Find the production levels if production is increased by 20%.

60. ***Manufacturing*** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. The number of units of vehicle i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}.$$

Find the production levels if production is increased by 10%.

61. ***Agriculture*** A fruit grower raises two crops, apples and peaches. Each of these crops is sent to three different outlets for sale. These outlets are The Farmer's Market, The Fruit Stand, and The Fruit Farm. The numbers of bushels of apples sent to the three outlets are 125, 100, and 75, respectively. The numbers of bushels of peaches sent to the three outlets are 100, 175, and 125, respectively. The profit per bushel for apples is \$3.50 and the profit per bushel for peaches is \$6.00.

(a) Write a matrix A that represents the number of bushels of each crop i that are shipped to each outlet j. State what each entry a_{ij} of the matrix represents.

(b) Write a matrix B that represents the profit per bushel of each fruit. State what each entry b_{ij} of the matrix represents.

(c) Find the product BA and state what each entry of the matrix represents.

62. ***Revenue*** A manufacturer of electronics produces three models of portable CD players, which are shipped to two warehouses. The number of units of model i that are shipped to warehouse j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 5{,}000 & 4{,}000 \\ 6{,}000 & 10{,}000 \\ 8{,}000 & 5{,}000 \end{bmatrix}.$$

The prices per unit are represented by the matrix

$$B = [\$39.50 \quad \$44.50 \quad \$56.50].$$

Compute BA and interpret the result.

63. ***Inventory*** A company sells five models of computers through three retail outlets. The inventories are represented by S.

$$S = \overset{\text{Model}}{\overbrace{\begin{matrix} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{matrix}}} \\ S = \begin{bmatrix} 3 & 2 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 & 3 \\ 4 & 2 & 1 & 3 & 2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \Bigg\} \text{ Outlet}$$

The wholesale and retail prices are represented by T.

$$T = \overset{\text{Price}}{\overbrace{\begin{matrix} \text{Wholesale} & \text{Retail} \end{matrix}}} \\ T = \begin{bmatrix} \$840 & \$1100 \\ \$1200 & \$1350 \\ \$1450 & \$1650 \\ \$2650 & \$3000 \\ \$3050 & \$3200 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{matrix} \Bigg\} \text{ Model}$$

Compute ST and interpret the result.

64. *Voting Preferences* The matrix

$$P = \overset{\text{From}}{\begin{matrix} \text{R} & \text{D} & \text{I} \end{matrix}} \atop \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \begin{matrix} \text{R} \\ \text{D} \\ \text{I} \end{matrix} \Bigg\} \text{ To}$$

is called a *stochastic matrix.* Each entry p_{ij} $(i \neq j)$ represents the proportion of the voting population that changes from party i to party j, and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Compute and interpret P^2.

65. *Voting Preferences* Use a graphing utility to find P^3, P^4, P^5, P^6, P^7, and P^8 for the matrix given in Exercise 64. Can you detect a pattern as P is raised to higher powers?

66. *Labor/Wage Requirements* A company that manufactures boats has the following labor-hour and wage requirements.

Labor per boat

Department: Cutting, Assembly, Packaging

$$S = \begin{bmatrix} 1.0 \text{ hr} & 0.5 \text{ hr} & 0.2 \text{ hr} \\ 1.6 \text{ hr} & 1.0 \text{ hr} & 0.2 \text{ hr} \\ 2.5 \text{ hr} & 2.0 \text{ hr} & 1.4 \text{ hr} \end{bmatrix} \begin{matrix} \text{Small} \\ \text{Medium} \\ \text{Large} \end{matrix} \Bigg\} \text{ Boat size}$$

Wages per hour

Plant: A, B

$$T = \begin{bmatrix} \$12 & \$10 \\ \$9 & \$8 \\ \$8 & \$7 \end{bmatrix} \begin{matrix} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{matrix} \Bigg\} \text{ Department}$$

Compute ST and interpret the result.

67. *Profit* At a local dairy mart, the numbers of gallons of skim milk, 2% milk, and whole milk sold over the weekend are represented by A.

Skim milk, 2% milk, Whole milk

$$A = \begin{bmatrix} 40 & 64 & 52 \\ 60 & 82 & 76 \\ 76 & 96 & 84 \end{bmatrix} \begin{matrix} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{matrix}$$

The selling prices (in dollars per gallon) and the profits (in dollars per gallon) for the three types of milk sold by the dairy mart are represented by B.

Selling price, Profit

$$B = \begin{bmatrix} 2.65 & 0.65 \\ 2.85 & 0.70 \\ 3.05 & 0.85 \end{bmatrix} \begin{matrix} \text{Skim milk} \\ \text{2\% milk} \\ \text{Whole milk} \end{matrix}$$

(a) Compute AB and interpret the result.

(b) Find the dairy mart's total profit from milk sales for the weekend.

68. *Profit* At a convenience store, the numbers of gallons of 87-octane, 89-octane, and 93-octane gasoline sold over the weekend are represented by A.

Octane: 87, 89, 93

$$A = \begin{bmatrix} 580 & 840 & 320 \\ 560 & 420 & 160 \\ 860 & 1020 & 540 \end{bmatrix} \begin{matrix} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{matrix}$$

The selling prices per gallon and the profits per gallon for the three grades of gasoline sold by the convenience store are represents by B.

Selling price, Profit

$$B = \begin{bmatrix} 1.95 & 0.32 \\ 2.05 & 0.36 \\ 2.15 & 0.40 \end{bmatrix} \begin{matrix} 87 \\ 89 \\ 93 \end{matrix} \Bigg\} \text{ Octane}$$

(a) Compute AB and interpret the result.

(b) Find the convenience store's profit from gasoline sales for the weekend.

69. *Exercise* The numbers of calories burned by individuals of different body weights performing different types of aerobic exercises for a 20-minute time period are shown in matrix A.

Calories burned: 120-lb person, 150-lb person

$$A = \begin{bmatrix} 109 & 136 \\ 127 & 159 \\ 64 & 79 \end{bmatrix} \begin{matrix} \text{Bicycling} \\ \text{Jogging} \\ \text{Walking} \end{matrix}$$

(a) A 120-pound person and a 150-pound person bicycled for 40 minutes, jogged for 10 minutes, and walked for 60 minutes. Organize the time spent exercising in a matrix B.

(b) Compute BA and interpret the result.

Model It

70. *Health Care* The health care plans offered this year by a local manufacturing plant are as follows. For individuals, the comprehensive plan costs \$694.32, the HMO standard plan costs \$451.80, and the HMO Plus plan costs \$489.48. For families, the comprehensive plan costs \$1725.36, the HMO standard plan costs \$1187.76 and the HMO Plus plan costs \$1248.12. The plant expects the costs of the plans to change next year as follows. For individuals, the costs for the comprehensive, HMO standard, and HMO Plus plans will be \$683.91, \$463.10, and \$499.27, respectively. For families, the costs for the comprehensive, HMO standard, and HMO Plus plans will be \$1699.48, \$1217.45, and \$1273.08, respectively.

(a) Organize the information using two matrices A and B, where A represents the health care plan costs for this year and B represents the health care plan costs for next year. State what each entry of each matrix represents.

(b) Compute $A - B$ and interpret the result.

(c) The employees receive monthly paychecks from which the health care plan costs are deducted. Use the matrices from part (a) to write matrices that show how much will be deducted from each employees' paycheck this year and next year.

(d) Suppose the costs of each plan instead increase by 4% next year. Write a matrix that shows the new monthly payment.

Synthesis

True or False? **In Exercises 71 and 72, determine whether the statement is true or false. Justify your answer.**

71. Two matrices can be added only if they have the same order.

72. $\begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix}$

Think About It **In Exercises 73–80, let matrices A, B, C, and D be of orders 2×3, 2×3, 3×2, and 2×2, respectively. Determine whether the matrices are of proper order to perform the operation(s). If so, give the order of the answer.**

73. $A + 2C$

74. $B - 3C$

75. AB

76. BC

77. $BC - D$

78. $CB - D$

79. $D(A - 3B)$

80. $(BC - D)A$

81. *Think About It* If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are nonzero matrices such that $AC = BC$, then A is *not* necessarily equal to B. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

82. *Think About It* If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = O$, it is *not* necessarily true that $A = O$ or $B = O$. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

83. *Exploration* Let A and B be unequal diagonal matrices of the same order. (A **diagonal matrix** is a square matrix in which each entry not on the main diagonal is zero.) Determine the products AB for several pairs of such matrices. Make a conjecture about a quick rule for such products.

84. *Exploration* Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(a) Find A^2, A^3, and A^4. Identify any similarities with i^2, i^3, and i^4.

(b) Find and identify B^2.

Skills Review

In Exercises 85–90, solve the equation.

85. $3x^2 + 20x - 32 = 0$

86. $8x^2 - 10x - 3 = 0$

87. $4x^3 + 10x^2 - 3x = 0$

88. $3x^3 + 22x^2 - 45x = 0$

89. $3x^3 - 12x^2 + 5x - 20 = 0$

90. $2x^3 - 5x^2 - 12x + 30 = 0$

In Exercises 91–94, solve the system of linear equations both graphically and algebraically.

91. $\begin{cases} -x + 4y = -9 \\ 5x - 8y = 39 \end{cases}$

92. $\begin{cases} 8x - 3y = -17 \\ -6x + 7y = 27 \end{cases}$

93. $\begin{cases} -x + 2y = -5 \\ -3x - y = -8 \end{cases}$

94. $\begin{cases} 6x - 13y = 11 \\ 9x + 5y = 41 \end{cases}$

8.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. In a ________ matrix, the number of rows equals the number of columns.
2. If there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$, then A^{-1} is called the ________ of A.
3. If a matrix A has an inverse, it is called invertible or ________; if it does not have an inverse, it is called ________.
4. If A is an invertible matrix, the system of linear equations represented by $AX = B$ has a unique solution given by $X =$ ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, show that B is the inverse of A.

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

6. $A = \begin{bmatrix} -4 & 1 & 5 \\ -1 & 2 & 4 \\ 0 & -1 & -1 \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{4} & -1 & -\frac{11}{4} \\ -\frac{1}{4} & 1 & \frac{7}{4} \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 4 & -1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & -1 & -1 \\ -4 & 9 & -5 & -6 \\ 0 & 1 & -1 & -1 \\ 3 & -5 & 3 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & -1 & -3 & 0 \\ -2 & -1 & 0 & -2 \\ 0 & 1 & 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -3 & -3 & 1 & -2 \\ 12 & 14 & -5 & 10 \\ -5 & -6 & 2 & -4 \\ -3 & -4 & 1 & -3 \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}$

In Exercises 11–26, find the inverse of the matrix (if it exists).

11. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

14. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 11 & 1 \\ -1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

20. $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

25. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

In Exercises 27–38, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

27. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

28. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

30. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ -4 & 4 & 3 \end{bmatrix}$

31. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

32. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

33. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

34. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

36. $\begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

37. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

38. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

In Exercises 39–44, use the formula on page 606 to find the inverse of the 2×2 matrix (if it exists).

39. $\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 7 & 12 \\ -8 & -5 \end{bmatrix}$

41. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

42. $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

43. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

44. $\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

In Exercises 45–48, use the inverse matrix found in Exercise 13 to solve the system of linear equations.

45. $\begin{cases} x - 2y = 5 \\ 2x - 3y = 10 \end{cases}$

46. $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

47. $\begin{cases} x - 2y = 4 \\ 2x - 3y = 2 \end{cases}$

48. $\begin{cases} x - 2y = 1 \\ 2x - 3y = -2 \end{cases}$

In Exercises 49 and 50, use the inverse matrix found in Exercise 21 to solve the system of linear equations.

49. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$

50. $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

In Exercises 51 and 52, use the inverse matrix found in Exercise 38 to solve the system of linear equations.

51. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

52. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

In Exercises 53–60, use an inverse matrix to solve (if possible) the system of linear equations.

53. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$

54. $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

55. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$

56. $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

57. $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$

58. $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{7}{2}y = -51 \end{cases}$

59. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

60. $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

In Exercises 61–66, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

61. $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$

62. $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

63. $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

64. $\begin{cases} -8x + 7y - 10z = -151 \\ 12x + 3y - 5z = 86 \\ 15x - 9y + 2z = 187 \end{cases}$

65. $\begin{cases} 7x - 3y + 2w = 41 \\ -2x + y - w = -13 \\ 4x + z - 2w = 12 \\ -x + y - w = -8 \end{cases}$

66. $\begin{cases} 2x + 5y + w = 11 \\ x + 4y + 2z - 2w = -7 \\ 2x - 2y + 5z + w = 3 \\ x - 3w = -1 \end{cases}$

Investment Portfolio **In Exercises 67–70, consider a person who invests in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 6.5% on AAA bonds, 7% on A bonds, and 9% on B bonds. The person invests twice as much in B bonds as in A bonds. Let x, y, and z represent the amounts invested in AAA, A, and B bonds, respectively.**

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.065x + 0.07y + 0.09z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

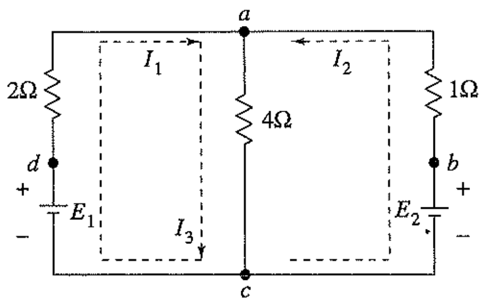
Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond.

	Total Investment	*Annual Return*
67.	$10,000	$705
68.	$10,000	$760
69.	$12,000	$835
70.	$500,000	$38,000

71. ***Circuit Analysis*** Consider the circuit shown in the figure. The currents I_1, I_2, and I_3, in amperes, are the solution of the system of linear equations

$$\begin{cases} 2I_1 \qquad + 4I_3 = E_1 \\ \qquad I_2 + 4I_3 = E_2 \\ I_1 + I_2 - \ I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the voltages.



(a) $E_1 = 14$ volts, $E_2 = 28$ volts

(b) $E_1 = 24$ volts, $E_2 = 23$ volts

Model It

72. ***Data Analysis: Licensed Drivers*** The table shows the numbers y (in millions) of licensed drivers in the United States for selected years 1997 to 2001. (Source: U.S. Federal Highway Administration)

Year	Drivers, y
1997	182.7
1999	187.2
2001	191.3

(a) Use the technique demonstrated in Exercises 57–62 in Section 7.2 to create a system of linear equations for the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the matrix capabilities of a graphing utility to find an inverse matrix to solve the system from part (a) and find the least squares regression line $y = at + b$.

(c) Use the result of part (b) to estimate the number of licensed drivers in 2003.

(d) The actual number of licensed drivers in 2003 was 196.2 million. How does this value compare with your estimate from part (c)?

Model It (continued)

(e) Use the result of part (b) to estimate when the number of licensed drivers will reach 208 million.

Synthesis

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. Multiplication of an invertible matrix and its inverse is commutative.

74. If you multiply two square matrices and obtain the identity matrix, you can assume that the matrices are inverses of one another.

75. If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, verify that the inverse is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

76. ***Exploration*** Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Write a 2×2 matrix and a 3×3 matrix in the form of A. Find the inverse of each.

(b) Use the result of part (a) to make a conjecture about the inverses of matrices in the form of A.

Skills Review

In Exercises 77 and 78, solve the inequality and sketch the solution on the real number line.

77. $|x + 7| \geq 2$

78. $|2x - 1| < 3$

In Exercises 79–82, solve the equation. Approximate the result to three decimal places.

79. $3^{x/2} = 315$

80. $2000e^{-x/5} = 400$

81. $\log_2 x - 2 = 4.5$

82. $\ln x + \ln(x - 1) = 0$

83. **Make a Decision** To work an extended application analyzing the number of U.S. households with color televisions from 1985 to 2005, visit this text's website at *college.hmco.com.* *(Data Source: Nielsen Media Research)*

8.4 The Determinant of a Square Matrix

What you should learn

- Find the determinants of 2×2 matrices.
- Find minors and cofactors of square matrices.
- Find the determinants of square matrices.

Why you should learn it

Determinants are often used in other branches of mathematics. For instance, Exercises 79–84 on page 618 show some types of determinants that are useful when changes in variables are made in calculus.

The Determinant of a 2×2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominators of the two fractions are the same. This denominator is called the *determinant* of the coefficient matrix of the system.

Coefficient Matrix *Determinant*

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad \det(A) = a_1b_2 - a_2b_1$$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2×2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a 2×2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

8.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. Both det(A) and $|A|$ represent the ________ of the matrix A.
2. The ________ M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of the square matrix A.
3. The ________ C_{ij} of the entry a_{ij} of the square matrix A is given by $(-1)^{i+j} M_{ij}$.
4. The method of finding the determinant of a matrix of order 2×2 or greater is called ________ by ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–16, find the determinant of the matrix.

1. $[5]$
2. $[-8]$
3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$
4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$
5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$
6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$
7. $\begin{bmatrix} -7 & 0 \\ 3 & 0 \end{bmatrix}$
8. $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$
9. $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$
10. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$
11. $\begin{bmatrix} -3 & -2 \\ -6 & -1 \end{bmatrix}$
12. $\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix}$
13. $\begin{bmatrix} 9 & 0 \\ 7 & 8 \end{bmatrix}$
14. $\begin{bmatrix} 0 & 6 \\ -3 & 2 \end{bmatrix}$
15. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{bmatrix}$
16. $\begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -1 & -\frac{1}{3} \end{bmatrix}$

In Exercises 17–22, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

17. $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$
18. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$
19. $\begin{bmatrix} 0.9 & 0.7 & 0 \\ -0.1 & 0.3 & 1.3 \\ -2.2 & 4.2 & 6.1 \end{bmatrix}$
20. $\begin{bmatrix} 0.1 & 0.1 & -4.3 \\ 7.5 & 6.2 & 0.7 \\ 0.3 & 0.6 & -1.2 \end{bmatrix}$
21. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{bmatrix}$
22. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 23–30, find all (a) minors and (b) cofactors of the matrix.

23. $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$
24. $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$
25. $\begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix}$
26. $\begin{bmatrix} -6 & 5 \\ 7 & -2 \end{bmatrix}$
27. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$
28. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}$
29. $\begin{bmatrix} 3 & -2 & 8 \\ 3 & 2 & -6 \\ -1 & 3 & 6 \end{bmatrix}$
30. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

In Exercises 31–36, find the determinant of the matrix by the method of expansion by cofactors. Expand using the indicated row or column.

31. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$
    (a) Row 1
    (b) Column 2
32. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$
    (a) Row 2
    (b) Column 3
33. $\begin{bmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{bmatrix}$
    (a) Row 2
    (b) Column 2
34. $\begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}$
    (a) Row 3
    (b) Column 1
35. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$
    (a) Row 2
    (b) Column 2
36. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$
    (a) Row 3
    (b) Column 1

In Exercises 37–52, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.

37. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$
38. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

39. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

41. $\begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 5 & 1 & 5 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

44. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

45. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

46. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

47. $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

48. $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

49. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

51. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

52. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 53–60, use the matrix capabilities of a graphing utility to evaluate the determinant.

53. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$

54. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

55. $\begin{vmatrix} 7 & 0 & -14 \\ -2 & 5 & 4 \\ -6 & 2 & 12 \end{vmatrix}$

56. $\begin{vmatrix} 3 & 0 & 0 \\ -2 & 5 & 0 \\ 12 & 5 & 7 \end{vmatrix}$

57. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$

58. $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

59. $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$

60. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

In Exercises 61–68, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.

61. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

62. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

63. $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

64. $A = \begin{bmatrix} 5 & 4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ 1 & -2 \end{bmatrix}$

65. $A = \begin{bmatrix} 0 & 1 & 2 \\ -3 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

66. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

67. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

68. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

In Exercises 69–74, evaluate the determinant(s) to verify the equation.

69. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

70. $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

71. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

72. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

73. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y)$

74. $\begin{vmatrix} a + b & a & a \\ a & a + b & a \\ a & a & a + b \end{vmatrix} = b^2(3a + b)$

In Exercises 75–78, solve for x.

75. $\begin{vmatrix} x - 1 & 2 \\ 3 & x - 2 \end{vmatrix} = 0$

76. $\begin{vmatrix} x - 2 & -1 \\ -3 & x \end{vmatrix} = 0$

77. $\begin{vmatrix} x + 3 & 2 \\ 1 & x + 2 \end{vmatrix} = 0$

78. $\begin{vmatrix} x + 4 & -2 \\ 7 & x - 5 \end{vmatrix} = 0$

In Exercises 79–84, evaluate the determinant in which the entries are functions. Determinants of this type occur when changes in variables are made in calculus.

79. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$

80. $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

81. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

82. $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

83. $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$

84. $\begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix}$

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. If a square matrix has an entire row of zeros, the determinant will always be zero.

86. If two columns of a square matrix are the same, the determinant of the matrix will be zero.

87. ***Exploration*** Find square matrices A and B to demonstrate that $|A + B| \neq |A| + |B|$.

88. ***Exploration*** Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

(a) Use a graphing utility to evaluate the determinants of four matrices of this type. Make a conjecture based on the results.

(b) Verify your conjecture.

89. ***Writing*** Write a brief paragraph explaining the difference between a square matrix and its determinant.

90. ***Think About It*** If A is a matrix of order 3×3 such that $|A| = 5$, is it possible to find $|2A|$? Explain.

Properties of Determinants **In Exercises 91–93, a property of determinants is given (A and B are square matrices). State how the property has been applied to the given determinants and use a graphing utility to verify the results.**

91. If B is obtained from A by interchanging two rows of A or interchanging two columns of A, then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

92. If B is obtained from A by adding a multiple of a row of A to another row of A or by adding a multiple of a column of A to another column of A, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

93. If B is obtained from A by multiplying a row by a nonzero constant c or by multiplying a column by a nonzero constant c, then $|B| = c|A|$.

(a) $\begin{vmatrix} 5 & 10 \\ 2 & -3 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 3 \end{vmatrix} = 12\begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

94. ***Exploration*** A **diagonal matrix** is a square matrix with all zero entries above and below its main diagonal. Evaluate the determinant of each diagonal matrix. Make a conjecture based on your results.

(a) $\begin{vmatrix} 7 & 0 \\ 0 & 4 \end{vmatrix}$

(b) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

Skills Review

In Exercises 95–100, find the domain of the function.

95. $f(x) = x^3 - 2x$

96. $g(x) = \sqrt[3]{x}$

97. $h(x) = \sqrt{16 - x^2}$

98. $A(x) = \dfrac{3}{36 - x^2}$

99. $g(t) = \ln(t - 1)$

100. $f(s) = 625e^{-0.5s}$

In Exercises 101 and 102, sketch the graph of the solution of the system of inequalities.

101. $\begin{cases} x + y \leq 8 \\ x \geq -3 \\ 2x - y < 5 \end{cases}$

102. $\begin{cases} -x - y > 4 \\ y \leq 1 \\ 7x + 4y \leq -10 \end{cases}$

In Exercises 103–106, find the inverse of the matrix (if it exists).

103. $\begin{bmatrix} -4 & 1 \\ 8 & -1 \end{bmatrix}$

104. $\begin{bmatrix} -5 & -8 \\ 3 & 6 \end{bmatrix}$

105. $\begin{bmatrix} -7 & 2 & 9 \\ 2 & -4 & -6 \\ 3 & 5 & 2 \end{bmatrix}$

106. $\begin{bmatrix} -6 & 2 & 0 \\ 1 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$

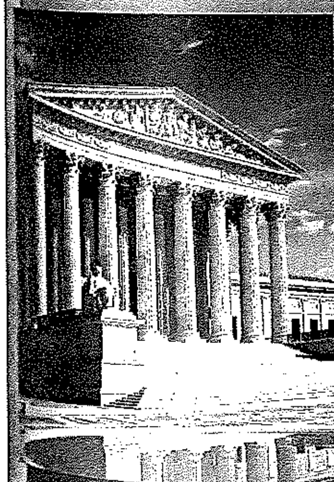
8.5 Applications of Matrices and Determinants

What you should learn

- Use Cramer's Rule to solve systems of linear equations.
- Use determinants to find the areas of triangles.
- Use a determinant to test for collinear points and find an equation of a line passing through two points.
- Use matrices to encode and decode messages.

Why you should learn it

You can use Cramer's Rule to solve real-life problems. For instance, in Exercise 58 on page 630, Cramer's Rule is used to find a quadratic model for the number of U.S. Supreme Court cases waiting to be tried.



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Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination with equations, and elimination with matrices. In this section, you will study one more method, **Cramer's Rule,** named after Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 8.4. There, it was pointed out that the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominator for x and y is simply the determinant of the *coefficient* matrix of the system. This determinant is denoted by D. The numerators for x and y are denoted by D_x and D_y, respectively. They are formed by using the column of constants as replacements for the coefficients of x and y, as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, D, D_x, and D_y are as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$	$\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$

8.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The method of using determinants to solve a system of linear equations is called ________ ________.
2. Three points are ________ if the points lie on the same line.
3. The area A of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by ________.
4. A message written according to a secret code is called a ________.
5. To encode a message, choose an invertible matrix A and multiply the ________ row matrices by A (on the right) to obtain ________ row matrices.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, use Cramer's Rule to solve (if possible) the system of equations.

1. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$

2. $\begin{cases} -4x - 7y = 47 \\ -x + 6y = -27 \end{cases}$

3. $\begin{cases} 3x + 2y = -2 \\ 6x + 4y = 4 \end{cases}$

4. $\begin{cases} 6x - 5y = 17 \\ -13x + 3y = -76 \end{cases}$

5. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 0.2x + 0.3y = 2.2 \end{cases}$

6. $\begin{cases} 2.4x - 1.3y = 14.63 \\ -4.6x + 0.5y = -11.51 \end{cases}$

7. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

8. $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

9. $\begin{cases} x + 2y + 3z = -3 \\ -2x + y - z = 6 \\ 3x - 3y + 2z = -11 \end{cases}$

10. $\begin{cases} 5x - 4y + z = -14 \\ -x + 2y - 2z = 10 \\ 3x + y + z = 1 \end{cases}$

In Exercises 11–14, use a graphing utility and Cramer's Rule to solve (if possible) the system of equations.

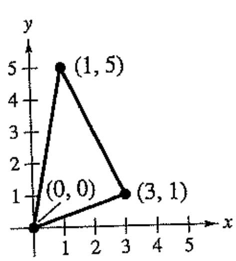
11. $\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 2 \\ 5x + 9y + 17z = 4 \end{cases}$

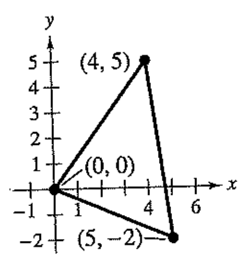
12. $\begin{cases} x + 2y - z = -7 \\ 2x - 2y - 2z = -8 \\ -x + 3y + 4z = 8 \end{cases}$

13. $\begin{cases} 2x + y + 2z = 6 \\ -x + 2y - 3z = 0 \\ 3x + 2y - z = 6 \end{cases}$

14. $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

In Exercises 15–24, use a determinant and the given vertices of a triangle to find the area of the triangle.

15.


16.


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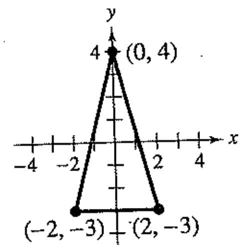
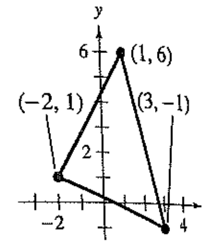
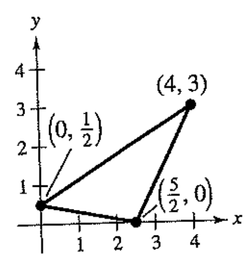
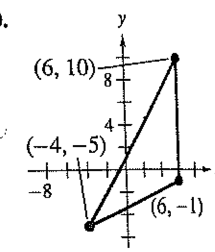


18.


19.


20.


21. $(-2, 4), (2, 3), (-1, 5)$
22. $(0, -2), (-1, 4), (3, 5)$
23. $(-3, 5), (2, 6), (3, -5)$
24. $(-2, 4), (1, 5), (3, -2)$

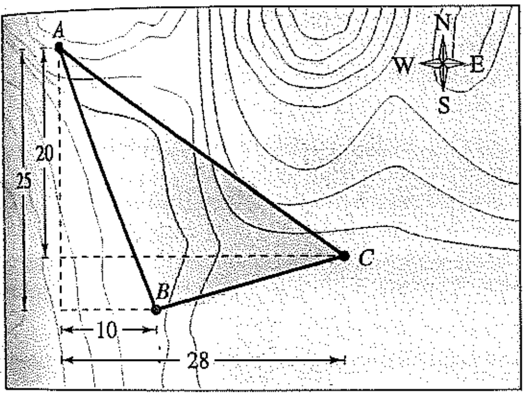
In Exercises 25 and 26, find a value of y such that the triangle with the given vertices has an area of 4 square units.

25. $(-5, 1), (0, 2), (-2, y)$
26. $(-4, 2), (-3, 5), (-1, y)$

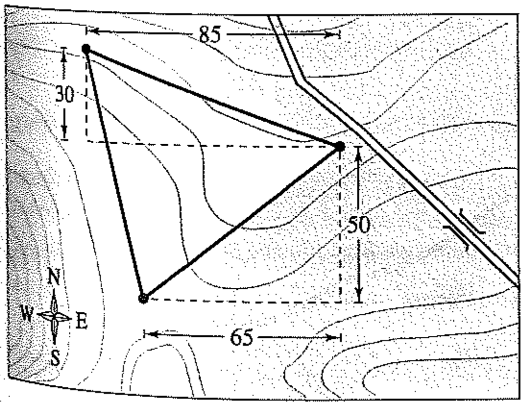
In Exercises 27 and 28, find a value of y such that the triangle with the given vertices has an area of 6 square units.

27. $(-2, -3), (1, -1), (-8, y)$
28. $(1, 0), (5, -3), (-3, y)$

29. ***Area of a Region*** A large region of forest has been infested with gypsy moths. The region is roughly triangular, as shown in the figure. From the northernmost vertex A of the region, the distances to the other vertices are 25 miles south and 10 miles east (for vertex B), and 20 miles south and 28 miles east (for vertex C). Use a graphing utility to approximate the number of square miles in this region.



30. ***Area of a Region*** You own a triangular tract of land, as shown in the figure. To estimate the number of square feet in the tract, you start at one vertex, walk 65 feet east and 50 feet north to the second vertex, and then walk 85 feet west and 30 feet north to the third vertex. Use a graphing utility to determine how many square feet there are in the tract of land.



In Exercises 31–36, use a determinant to determine whether the points are collinear.

31. $(3, -1), (0, -3), (12, 5)$ **32.** $(-3, -5), (6, 1), (10, 2)$

33. $(2, -\frac{1}{2}), (-4, 4), (6, -3)$ **34.** $(0, 1), (4, -2), (-2, \frac{5}{2})$

35. $(0, 2), (1, 2.4), (-1, 1.6)$ **36.** $(2, 3), (3, 3.5), (-1, 2)$

In Exercises 37 and 38, find y such that the points are collinear.

37. $(2, -5), (4, y), (5, -2)$ **38.** $(-6, 2), (-5, y), (-3, 5)$

In Exercises 39–44, use a determinant to find an equation of the line passing through the points.

39. $(0, 0), (5, 3)$ **40.** $(0, 0), (-2, 2)$

41. $(-4, 3), (2, 1)$ **42.** $(10, 7), (-2, -7)$

43. $(-\frac{1}{2}, 3), (\frac{5}{2}, 1)$ **44.** $(\frac{2}{3}, 4), (6, 12)$

In Exercises 45 and 46, find the uncoded 1 × 3 row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
45. TROUBLE IN RIVER CITY	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
46. PLEASE SEND MONEY	$\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

In Exercises 47–50, write a cryptogram for the message using the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

47. CALL AT NOON

48. ICEBERG DEAD AHEAD

49. HAPPY BIRTHDAY

50. OPERATION OVERLOAD

In Exercises 51–54, use A^{-1} to decode the cryptogram.

51. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

11 21 64 112 25 50 29 53 23 46
40 75 55 92

52. $A = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$

−136 58 −173 72 −120 51 −95 38
−178 73 −70 28 −242 101 −115 47
−90 36 −115 49 −199 82

53. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

9 −1 −9 38 −19 −19 28 −9 −19 −80 25
41 −64 21 31 9 −5 −4

54. $A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}$

112 −140 83 19 −25 13 72 −76 61 95
−118 71 20 21 38 35 −23 36 42 −48 32

In Exercises 55 and 56, decode the cryptogram by using the inverse of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

55. 20 17 −15 −12 −56 −104 1 −25 −65 62 143 181

56. 13 −9 −59 61 112 106 −17 −73 −131 11 24 29 65 144 172

57. The following cryptogram was encoded with a 2×2 matrix.

8 21 −15 −10 −13 −13 5 10 5 25 5 19 −1 6 20 40 −18 −18 1 16

The last word of the message is _RON. What is the message?

Model It

58. ***Data Analysis: Supreme Court*** The table shows the numbers y of U.S. Supreme Court cases waiting to be tried for the years 2000 through 2002. (Source: Office of the Clerk, Supreme Court of the United States)

Year	Number of cases, y
2000	8965
2001	9176
2002	9406

(a) Use the technique demonstrated in Exercises 67–70 in Section 7.3 to create a system of linear equations for the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use Cramer's Rule to solve the system from part (a) and find the least squares regression parabola $y = at^2 + bt + c$.

(c) Use a graphing utility to graph the parabola from part (b).

(d) Use the graph from part (c) to estimate when the number of U.S. Supreme Court cases waiting to be tried will reach 10,000.

Synthesis

True or False? **In Exercises 59–61, determine whether the statement is true or false. Justify your answer.**

59. In Cramer's Rule, the numerator is the determinant of the coefficient matrix.

60. You cannot use Cramer's Rule when solving a system of linear equations if the determinant of the coefficient matrix is zero.

61. In a system of linear equations, if the determinant of the coefficient matrix is zero, the system has no solution.

62. ***Writing*** At this point in the text, you have learned several methods for solving systems of linear equations. Briefly describe which method(s) you find easiest to use and which method(s) you find most difficult to use.

Skills Review

In Exercises 63–66, use any method to solve the system of equations.

63. $\begin{cases} -x - 7y = -22 \\ 5x + y = -26 \end{cases}$

64. $\begin{cases} 3x + 8y = 11 \\ -2x + 12y = -16 \end{cases}$

65. $\begin{cases} -x - 3y + 5z = -14 \\ 4x + 2y - z = -1 \\ 5x - 3y + 2z = -11 \end{cases}$

66. $\begin{cases} 5x - y - z = 7 \\ -2x + 3y + z = -5 \\ 4x + 10y - 5z = -37 \end{cases}$

In Exercises 67 and 68, sketch the region determined by the constraints. Then find the minimum and maximum values of the objective function and where they occur, subject to the constraints.

67. Objective function:

$z = 6x + 4y$

Constraints:

$x \geq 0$

$y \geq 0$

$x + 6y \leq 30$

$6x + y \leq 40$

68. Objective function:

$z = 6x + 7y$

Constraints:

$x \geq 0$

$y \geq 0$

$4x + 3y \geq 24$

$x + 3y \geq 15$

8 Review Exercises

8.1 **In Exercises 1–4, determine the order of the matrix.**

1. $\begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -1 & 0 & 6 \\ -2 & 7 & 1 & 4 \end{bmatrix}$

3. $[3]$

4. $[6 \quad 2 \quad -5 \quad 8 \quad 0]$

In Exercises 5 and 6, write the augmented matrix for the system of linear equations.

5. $\begin{cases} 3x - 10y = 15 \\ 5x + 4y = 22 \end{cases}$

6. $\begin{cases} 8x - 7y + 4z = 12 \\ 3x - 5y + 2z = 20 \\ 5x + 3y - 3z = 26 \end{cases}$

In Exercises 7 and 8, write the system of linear equations represented by the augmented matrix. (Use variables *x*, *y*, *z*, and *w*, if applicable.)

7. $\left[\begin{array}{ccc:c} 5 & 1 & 7 & -9 \\ 4 & 2 & 0 & 10 \\ 9 & 4 & 2 & 3 \end{array}\right]$

8. $\left[\begin{array}{cccc:c} 13 & 16 & 7 & 3 & 2 \\ 1 & 21 & 8 & 5 & 12 \\ 4 & 10 & -4 & 3 & -1 \end{array}\right]$

In Exercises 9 and 10, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.

9. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 8 & 16 \\ 3 & -1 & 2 \\ -2 & 10 & 12 \end{bmatrix}$

In Exercises 11–14, write the system of linear equations represented by the augmented matrix. Then use back-substitution to solve the system. (Use variables *x*, *y*, and *z*.)

11. $\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccc:c} 1 & 3 & -9 & 4 \\ 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$

13. $\left[\begin{array}{ccc:c} 1 & -5 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

14. $\left[\begin{array}{ccc:c} 1 & -8 & 0 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

In Exercises 15–24, use matrices and Gaussian elimination with back-substitution to solve the system of equations (if possible).

15. $\begin{cases} 5x + 4y = 2 \\ -x + y = -22 \end{cases}$

16. $\begin{cases} 2x - 5y = 2 \\ 3x - 7y = 1 \end{cases}$

17. $\begin{cases} 0.3x - 0.1y = -0.13 \\ 0.2x - 0.3y = -0.25 \end{cases}$

18. $\begin{cases} 0.2x - 0.1y = 0.07 \\ 0.4x - 0.5y = -0.01 \end{cases}$

19. $\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$

20. $\begin{cases} 2x + 3y + 3z = 3 \\ 6x + 6y + 12z = 13 \\ 12x + 9y - z = 2 \end{cases}$

21. $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$

22. $\begin{cases} x + 2y + 6z = 1 \\ 2x + 5y + 15z = 4 \\ 3x + y + 3z = -6 \end{cases}$

23. $\begin{cases} 2x + y + z = 6 \\ -2y + 3z - w = 9 \\ 3x + 3y - 2z - 2w = -11 \\ x + z + 3w = 14 \end{cases}$

24. $\begin{cases} x + 2y + w = 3 \\ -3y + 3z = 0 \\ 4x + 4y + z + 2w = 0 \\ 2x + z = 3 \end{cases}$

In Exercises 25–28, use matrices and Gauss-Jordan elimination to solve the system of equations.

25. $\begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$

26. $\begin{cases} 4x + 4y + 4z = 5 \\ 4x - 2y - 8z = 1 \\ 5x + 3y + 8z = 6 \end{cases}$

27. $\begin{cases} 2x - y + 9z = -8 \\ -x - 3y + 4z = -15 \\ 5x + 2y - z = 17 \end{cases}$

28. $\begin{cases} -3x + y + 7z = -20 \\ 5x - 2y - z = 34 \\ -x + y + 4z = -8 \end{cases}$

In Exercises 29 and 30, use the matrix capabilities of a graphing utility to reduce the augmented matrix corresponding to the system of equations, and solve the system.

29. $\begin{cases} 3x - y + 5z - 2w = -44 \\ x + 6y + 4z - w = 1 \\ 5x - y + z + 3w = -15 \\ 4y - z - 8w = 58 \end{cases}$

30. $\begin{cases} 4x + 12y + 2z = 20 \\ x + 6y + 4z = 12 \\ x + 6y + z = 8 \\ -2x - 10y - 2z = -10 \end{cases}$

In Exercises 31–34, find x and y.

31. $\begin{bmatrix} -1 & x \\ y & 9 \end{bmatrix} = \begin{bmatrix} -1 & 12 \\ -7 & 9 \end{bmatrix}$

32. $\begin{bmatrix} -1 & 0 \\ x & 5 \\ -4 & y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 8 & 5 \\ -4 & 0 \end{bmatrix}$

33. $\begin{bmatrix} x+3 & -4 & 4y \\ 0 & -3 & 2 \\ -2 & y+5 & 6x \end{bmatrix} = \begin{bmatrix} 5x-1 & -4 & 44 \\ 0 & -3 & 2 \\ -2 & 16 & 6 \end{bmatrix}$

34. $\begin{bmatrix} -9 & 4 & 2 & -5 \\ 0 & -3 & 7 & -4 \\ 6 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 4 & x-10 & -5 \\ 0 & -3 & 7 & 2y \\ \frac{1}{2}x & -1 & 1 & 0 \end{bmatrix}$

In Exercises 35–38, if possible, find (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $A + 3B$.

35. $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix}$

36. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \\ 15 & 30 \end{bmatrix}$

37. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 4 & 12 \\ 20 & 40 \end{bmatrix}$

38. $A = \begin{bmatrix} 6 & -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix}$

In Exercises 39–42, perform the matrix operations. If it is not possible, explain why.

39. $\begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 10 & -20 \\ 14 & -3 \end{bmatrix}$

40. $\begin{bmatrix} -11 & 16 & 19 \\ -7 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 8 & -4 \\ -2 & 10 \end{bmatrix}$

41. $-2\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8\begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$

42. $-\begin{bmatrix} 8 & -1 & 8 \\ -2 & 4 & 12 \\ 0 & -6 & 0 \end{bmatrix} - 5\begin{bmatrix} -2 & 0 & -4 \\ 3 & -1 & 1 \\ 6 & 12 & -8 \end{bmatrix}$

In Exercises 43 and 44, use the matrix capabilities of a graphing utility to evaluate the expression.

43. $3\begin{bmatrix} 8 & -2 & 5 \\ 1 & 3 & -1 \end{bmatrix} + 6\begin{bmatrix} 4 & -2 & -3 \\ 2 & 7 & 6 \end{bmatrix}$

44. $-5\begin{bmatrix} 2 & 0 \\ 7 & -2 \\ 8 & 2 \end{bmatrix} + 4\begin{bmatrix} 4 & -2 \\ 6 & 11 \\ -1 & 3 \end{bmatrix}$

In Exercises 45–48, solve for X in the equation given

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

45. $X = 3A - 2B$

46. $6X = 4A + 3B$

47. $3X + 2A = B$

48. $2A - 5B = 3X$

In Exercises 49–52, find AB, if possible.

49. $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix}$

50. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \\ 15 & 30 \end{bmatrix}$

51. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \end{bmatrix}$

52. $A = \begin{bmatrix} 6 & -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix}$

In Exercises 53–60, perform the matrix operations. If it is not possible, explain why.

53. $\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

54. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

55. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -2 & 0 \\ 8 & 0 \end{bmatrix}$

56. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

57. $\begin{bmatrix} 4 \\ 6 \end{bmatrix}\begin{bmatrix} 6 & -2 \end{bmatrix}$

8.5 **In Exercises 107–110, use Cramer's Rule to solve (if possible) the system of equations.**

107. $\begin{cases} 5x - 2y = 6 \\ -11x + 3y = -23 \end{cases}$

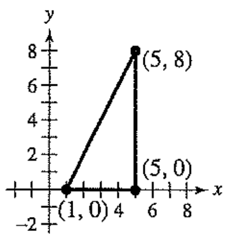
108. $\begin{cases} 3x + 8y = -7 \\ 9x - 5y = 37 \end{cases}$

109. $\begin{cases} -2x + 3y - 5z = -11 \\ 4x - y + z = -3 \\ -x - 4y + 6z = 15 \end{cases}$

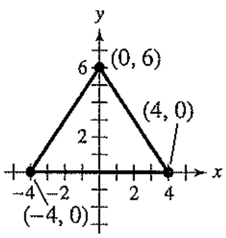
110. $\begin{cases} 5x - 2y + z = 15 \\ 3x - 3y - z = -7 \\ 2x - y - 7z = -3 \end{cases}$

In Exercises 111–114, use a determinant and the given vertices of a triangle to find the area of the triangle.

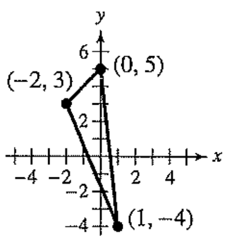
111.



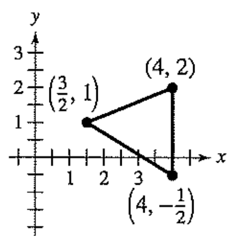
112.



113.



114.



In Exercises 115 and 116, use a determinant to determine whether the points are collinear.

115. $(-1, 7), (3, -9), (-3, 15)$

116. $(0, -5), (-2, -6), (8, -1)$

In Exercises 117–120, use a determinant to find an equation of the line passing through the points.

117. $(-4, 0), (4, 4)$

118. $(2, 5), (6, -1)$

119. $\left(-\frac{5}{2}, 3\right), \left(\frac{7}{2}, 1\right)$

120. $(-0.8, 0.2), (0.7, 3.2)$

In Exercises 121 and 122, find the uncoded 1 × 3 row matrices for the message. Then encode the message using the encoding matrix.

Message	*Encoding Matrix*
121. LOOK OUT BELOW	$\begin{bmatrix} 2 & -2 & 0 \\ 3 & 0 & -3 \\ -6 & 2 & 3 \end{bmatrix}$
122. RETURN TO BASE	$\begin{bmatrix} 2 & 1 & 0 \\ -6 & -6 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

In Exercises 123 and 124, decode the cryptogram by using the inverse of the matrix

$$A = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}.$$

123. −5 11 −2 370 −265 225 −57 48 −33 32 −15 20 245 −171 147

124. 145 −105 92 264 −188 160 23 −16 15 129 −84 78 −9 8 −5 159 −118 100 219 −152 133 370 −265 225 −105 84 −63

Synthesis

True or False? **In Exercises 125 and 126, determine whether the statement is true or false. Justify your answer.**

125. It is possible to find the determinant of a 4 × 5 matrix.

126. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_1 & a_{32} + c_2 & a_{33} + c_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_1 & c_2 & c_3 \end{vmatrix}$

127. Under what conditions does a matrix have an inverse?

128. ***Writing*** What is meant by the cofactor of an entry of a matrix? How are cofactors used to find the determinant of the matrix?

129. Three people were asked to solve a system of equations using an augmented matrix. Each person reduced the matrix to row-echelon form. The reduced matrices were

$$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 1 & \vdots & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Can all three be right? Explain.

130. ***Think About It*** Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has a unique solution.

131. Solve the equation for λ.

$$\begin{vmatrix} 2 - \lambda & 5 \\ 3 & -8 - \lambda \end{vmatrix} = 0$$